ON THE 3-TORSION PART OF THE HOMOLOGY OF THE CHESSBOARD COMPLEX

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ABSTRACT. Let $1 \le m \le n$. We prove various results about the chessboard complex $\mathsf{M}_{m,n}$, which is the simplicial complex of matchings in the complete bipartite graph $K_{m,n}$. First, we demonstrate that there is nonvanishing 3-torsion in $\tilde{H}_d(\mathsf{M}_{m,n};\mathbb{Z})$ whenever $\frac{m+n-4}{3} \le d \le m-4$ and whenever $6 \le m < n$ and $d = m-3$. Combining this result with theorems due to Friedman and Hanlon and to Shareshian and Wachs, we characterize all triples (m,n,d) satisfying $\tilde{H}_d(\mathsf{M}_{m,n};\mathbb{Z}) \neq 0$. Second, for each $k \ge 0$, we show that there is a polynomial $f_k(a,b)$ of degree $3k$ such that the dimension of $\tilde{H}_{k+a+2b-2}(\mathsf{M}_{k+a+3b-1,k+2a+3b-1};\mathbb{Z}_3)$, viewed as a vector space over $\mathbb{Z}_3$, is at most $f_k(a,b)$ for all $a \ge 0$ and $b \ge k+2$. Third, we give a computer-free proof that $\tilde{H}_2(\mathsf{M}_{5,5};\mathbb{Z}) \cong \mathbb{Z}_3$. Several proofs are based on a new long exact sequence relating the homology of a certain subcomplex of $\mathsf{M}_{m,n}$ to the homology of $\mathsf{M}_{m-2,n-1}$ and $\mathsf{M}_{m-2,n-3}$.

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1. Introduction

Given a family Δ of graphs on a fixed vertex set, we identify each member of Δ with its edge set. In particular, if Δ is closed under deletion of edges, then Δ is an abstract simplicial complex.

A *matching* in a simple graph G is a subset σ of the edge set of G such that no vertex appears in more than one edge in σ. Let $\mathsf{M}(G)$ be the family of matchings in G; $\mathsf{M}(G)$ is a simplicial complex. We write $\mathsf{M}_n = \mathsf{M}(K_n)$ and $\mathsf{M}_{m,n} = \mathsf{M}(K_{m,n})$, where K_n is the complete graph on the vertex set $[n] = \{1,\dots,n\}$ and $K_{m,n}$ is the complete bipartite graph with block sizes m and n. M_n is the *matching complex* and $\mathsf{M}_{m,n}$ is the *chessboard complex.*

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The topology of M_n, $\mathsf{M}_{m,n}$, and related complexes has been subject to analysis in a number of theses [1, 6, 9, 10, 15, 17] and papers [2, 3, 4, 5, 7, 8, 16, 18, 19, 22]; see Wachs [21] for an excellent survey and further references.

Despite the simplicity of the definition, the homology of the matching complex M_n and the chessboard complex $\mathsf{M}_{m,n}$ turns out to have a complicated structure. The rational homology is well-understood and easy to describe thanks to beautiful results due to Bouc [5] and Friedman and Hanlon [8], but very little is known about the integral homology and the homology over finite fields.

A previous paper [12] contains a number of results about the integral homology of the matching complex M_n. The purpose of the present paper is to extend a few of these results to the chessboard complex $\mathsf{M}_{m,n}$.

For $1 \le m \le n$, define

$$\nu_{m,n} = \min\{m-1, \lceil \tfrac{m+n-4}{3} \rceil\} = \begin{cases} \lceil \frac{m+n-4}{3} \rceil & \text{if } m \le n \le 2m+1; \\ m-1 & \text{if } n \ge 2m-1. \end{cases}$$

Note that $\lceil \frac{m+n-4}{3} \rceil = m-1$ for $2m-1 \le n \le 2m+1$. By a theorem due to Shareshian and Wachs [19], $\mathsf{M}_{m,n}$ contains nonvanishing homology in degree $\nu_{m,n}$ for all $m, n \ge 1$ except $(m,n) = (1,1)$. Previously, Friedman and Hanlon demonstrated that this bottom nonvanishing homology group is finite if and only if $m \le n \le 2m-5$ and $(m,n) \notin \{(6,6),(7,7),(8,9)\}$.

To settle their theorem, Shareshian and Wachs demonstrated that $\tilde{H}_{\nu_{m,n}}(\mathsf{M}_{m,n}; \mathbb{Z})$ contains nonvanishing 3-torsion whenever the group is finite. One of our main results provides upper bounds on the rank of the 3-torsion part. Specifically, in Section 4.2, we prove the following:

Theorem 1. *For each $k \ge 0$, $a \ge 0$, and $b \ge k+2$, we have that $\dim \tilde{H}_{k+a+2b-2}(\mathsf{M}_{k+a+3b-1,k+2a+3b-1}; \mathbb{Z}_3)$ is bounded by a polynomial in a and b of degree $3k$.*

An equivalent way of expressing Theorem 1 is to say that

$$\dim \tilde{H}_d(\mathsf{M}_{m,n}; \mathbb{Z}_3) \le f_{3d-m-n+4}(n-m, m-d-1)$$

whenever $m \le n \le 2m-5$ and $\frac{m+n-4}{3} \le d \le \frac{2m+n-7}{4}$, where f_k is a polynomial of degree $3k$ for each k. The bound in Theorem 1 remains true over any coefficient field.

Note that we express the theorem in terms of the following transformation:

(1)
$$\left\{\begin{array}{rcrcrcrcr} k & = & -m & - & n & + & 3d & + & 4 \\ a & = & -m & + & n & & & & \\ b & = & m & & & - & d & - & 1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcrcrcrcr} m & = & k & + & a & + & 3b & - & 1 \\ n & = & k & + & 2a & + & 3b & - & 1 \\ d & = & k & + & a & + & 2b & - & 2. \end{array}\right.$$

Assuming that $m \leq n$, each of the three new variables measures the difference between two important parameters:

- For $m \leq n \leq 2m+1$, we have that k measures the difference between the degree d and the bottom degree in which $\mathsf{M}_{m,n}$ has nonvanishing homology; $\frac{k}{3} = d - \frac{m+n-4}{3}$.
- a is the difference between the block sizes n and m.
- b is the difference between $\dim \mathsf{M}_{m,n} = m-1$ and d.

To establish Theorem 1, we introduce two new long exact sequences; see Sections 2.3 and 2.4. These two sequences involve the subcomplex $\Gamma_{m,n}$ of $\mathsf{M}_{m,n}$ obtained by fixing a vertex in the block of size n and removing all but two of the edges that are incident to this vertex. Our first sequence is very simple and relates the homology of $\mathsf{M}_{m,n}$ to that of $\Gamma_{m,n}$ and $\mathsf{M}_{m-1,n-1}$. Our second sequence is more complicated and relates $\Gamma_{m,n}$ to $\mathsf{M}_{m-2,n-1}$ and $\mathsf{M}_{m-2,n-3}$. Combining the two sequences and "cancelling out" $\Gamma_{m,n}$, we obtain a bound on the dimension of the $\mathbb{Z}_3$-homology of $\mathsf{M}_{m,n}$ in terms of $\mathsf{M}_{m-1,n-1}$, $\mathsf{M}_{m-2,n-1}$, and $\mathsf{M}_{m-2,n-3}$. By an induction argument, we obtain Theorem 1.

For $k=0$, Theorem 1 says that $\dim \tilde{H}_{\nu_{m,n}}(\mathsf{M}_{m,n};\mathbb{Z}_3)$ is bounded by a constant whenever $m \leq n \leq 2m-5$ and $m+n \equiv 1 \pmod 3$. Indeed, Shareshian and Wachs [19] proved that $\tilde{H}_{\nu_{m,n}}(\mathsf{M}_{m,n};\mathbb{Z}) \cong \mathbb{Z}_3$ for any m and n satisfying these equations. Their proof was by induction on $m+n$ and relied on a computer calculation of $\tilde{H}_2(\mathsf{M}_{5,5};\mathbb{Z})$. In Section 3, we provide a computer-free proof that $\tilde{H}_2(\mathsf{M}_{5,5};\mathbb{Z}) \cong \mathbb{Z}_3$, again using the exact sequences from Sections 2.3 and 2.4.

In Section 4.1, we use results about the matching complex M_n from a previous paper [12] to extend Shareshian and Wachs' 3-torsion result to higher-degree groups:

Theorem 2. *For $m+1 \leq n \leq 2m-5$, there is 3-torsion in $\tilde{H}_d(\mathsf{M}_{m,n};\mathbb{Z})$ whenever $\frac{m+n-4}{3} \leq d \leq m-3$. There is also 3-torsion in $\tilde{H}_d(\mathsf{M}_{m,m};\mathbb{Z})$ whenever $\frac{2m-4}{3} \leq d \leq m-4$.*

Note that $m+1 \leq n \leq 2m-5$ and $\frac{m+n-4}{3} \leq d \leq m-3$ if and only if $k \geq 0$, $a \geq 1$, and $b \geq 2$, where k, a, and b are defined as in (1). Moreover, $m=n$ and $\frac{2m-4}{3} \leq d \leq m-4$ if and only if $k \geq 0$, $a=0$, and $b \geq 3$.

Our proof of Theorem 2 relies on properties of the top homology group of $\mathsf{M}_{k,k+1}$ for different values of k; this group was of importance also in the work of Shareshian and Wachs [19].

Thanks to Theorem 2 and Friedman and Hanlon's formula for the rational homology [8], we may characterize those (d, m, n) satisfying $\tilde{H}_d(\mathsf{M}_{m,n}; \mathbb{Z}) \neq 0$:

Theorem 3. *For $1 \leq m \leq n$, we have that $\tilde{H}_d(\mathsf{M}_{m,n}; \mathbb{Z})$ is nonzero if and only if either of the following is true:*

- $\lceil \frac{m+n-4}{3} \rceil \leq d \leq m - 2$. *Equivalently, $k \geq 0$, $a \geq 0$, and $b \geq 1$.*
- $d = m - 1$ *and* $n \geq m + 1$. *Equivalently, $k \geq 2 - a$, $a \geq 1$, and $b = 0$.*

Again, see Section 4.1 for details.

The problem of detecting p-torsion in the homology of $\tilde{M}_{m,n}$ for $p \neq 3$ remains open. In this context, we may mention that there is 5-torsion in the homology of the matching complex M_{14} [13]. By computer calculations [14], further p-torsion is known to exist for $p \in \{5, 7, 11, 13\}$.

1.1. **Notation.** We identify the two parts of the graph $K_{m,n}$ with the two sets $[m] = \{1, 2, \ldots, m\}$ and $[\overline{n}] = \{\overline{1}, \overline{2}, \ldots, \overline{n}\}$. The latter set should be interpreted as a disjoint copy of $[n]$; hence each edge is of the form $i\overline{j}$, where $i \in [m]$ and $j \in [n]$. Sometimes, it will be useful to view $\mathsf{M}_{m,n}$ as a subcomplex of the matching complex M_{m+n} on the complete graph K_{m+n}. In such situations, we identify the vertex $\overline{j}$ in $K_{m,n}$ with the vertex $m + j$ in K_{m+n} for each $j \in [n]$.

For finite sets S and T, we let $\mathsf{M}_{S,T}$ denote the matching complex on the complete bipartite graph with blocks S and T, viewed as disjoint sets in the manner described above. In particular, $\mathsf{M}_{[m],[n]} = \mathsf{M}_{m,n}$. For integers $a \leq b$, we write $[a, b] = \{a, a + 1, \ldots, b - 1, b\}$.

The *join* of two families of sets Δ and Σ, assumed to be defined on disjoint ground sets, is the family $\Delta * \Sigma = \{\delta \cup \sigma : \delta \in \Delta, \sigma \in \Sigma\}$.

Whenever we discuss the homology of a simplicial complex or the relative homology of a pair of simplicial complexes, we mean reduced simplicial homology. For a simplicial complex Σ and a coefficient ring $\mathbb{F}$, we let $e_0 \wedge \cdots \wedge e_d$ denote a generator of $\tilde{C}_d(\Sigma; \mathbb{F})$ corresponding to the set $\{e_0, \ldots, e_d\} \in \Sigma$. Given a cycle z in a chain group $\tilde{C}_d(\Sigma; \mathbb{F})$, whenever we talk about z as an element in the induced homology group $\tilde{H}_d(\Sigma; \mathbb{F})$, we really mean the homology class of z.

We will often consider pairs of complexes (Γ, Δ) such that $\Gamma \setminus \Delta$ is a union of families of the form

$$\Sigma = \{\sigma\} * \mathsf{M}_{S,T},$$

where $\sigma = \{e_1, \ldots, e_s\}$ is a set of pairwise disjoint edges of the form $i\overline{j}$, and where S and T are subsets of $[m]$ and $[n]$, respectively, such that $S \cap e_i = \overline{T} \cap e_i = \emptyset$ for each i. We may write the chain complex of Σ as

$$\tilde{C}_d(\Sigma; \mathbb{F}) = (e_1 \wedge \cdots \wedge e_s)\mathbb{F} \otimes_{\mathbb{F}} \tilde{C}_{d-|\sigma|}(\mathsf{M}_{S,T}; \mathbb{F}),$$

defining the boundary operator as

$$\partial(e_1 \wedge \cdots \wedge e_s \otimes_{\mathbb{F}} c) = (-1)^s e_1 \wedge \cdots \wedge e_s \otimes_{\mathbb{F}} \partial(c).$$

For simplicity, we will often suppress $\mathbb{F}$ from notation. For example, by some abuse of notation, we will write

$$e_1 \wedge \cdots \wedge e_s \otimes \tilde{C}_{d-|\sigma|}(\mathsf{M}_{S,T}) = (e_1 \wedge \cdots \wedge e_s)\mathbb{F} \otimes_{\mathbb{F}} \tilde{C}_{d-|\sigma|}(\mathsf{M}_{S,T}; \mathbb{F}).$$

We say that a cycle z in $\tilde{C}_{d-1}(\mathsf{M}_{m,n}; \mathbb{F})$ has *type* $\left[\begin{smallmatrix} m_1,n_1 \\ d_1 \end{smallmatrix}\right] \wedge \cdots \wedge \left[\begin{smallmatrix} m_s,n_s \\ d_s \end{smallmatrix}\right]$ if there are partitions $[m] = \bigcup_{i=1}^s S_i$ and $[n] = \bigcup_{i=1}^s T_i$ such that $|S_i| = m_i$ and $|T_i| = n_i$ and such that $z = z_1 \wedge \cdots \wedge z_s$, where z_i is a cycle in $\tilde{C}_{d_i-1}(\mathsf{M}_{S_i,T_i}; \mathbb{F})$ for each i.

1.2. **Two classical results.** Before proceeding, we list two classical results pertaining to the topology of the chessboard complex $\mathsf{M}_{m,n}$.

Theorem 1.1 (Björner et al. [4])**.** *For $m, n \geq 1$, $\mathsf{M}_{m,n}$ is $(\nu_{m,n} - 1)$-connected.*

Indeed, the $\nu_{m,n}$-skeleton of $\mathsf{M}_{m,n}$ is vertex decomposable [22]. Garst [9] settled the case $n \geq 2m-1$ in Theorem 1.1. As already mentioned in the introduction, there is nonvanishing homology in degree $\nu_{m,n}$ for all $(m,n) \neq (1,1)$; see Section 3 for details.

The transformation (1) maps the set $\{(m,n,\nu_{m,n}) : 1 \leq m \leq n\}$ to the set of triples (k,a,b) satisfying either of the following:

- $k \in \{0,1,2\}$, $a \geq 0$, and $b \geq 1$ (corresponding to $d = \lceil \frac{m+n-4}{3} \rceil$ and $m \leq n \leq 2m-2$).
- $2-a \leq k \leq 2$ and $b = 0$ (corresponding to $0 \leq d = m-1$ and $n \geq 2m-1$).

Friedman and Hanlon [8] established a formula for the rational homology of $\mathsf{M}_{m,n}$; see Wachs [21] for an overview. For our purposes, the most important consequence is the following result:

Theorem 1.2 (Friedman and Hanlon [8])**.** *For $1 \leq m \leq n$, we have that $\tilde{H}_d(\mathsf{M}_{m,n}; \mathbb{Z})$ is infinite if and only if $(m-d-1)(n-d-1) \leq d+1$,*

$m \ge d+1$, and $n \ge d+2$. In particular, $\tilde{H}_{\nu_{m,n}}(\mathsf{M}_{m,n};\mathbb{Z})$ is finite if and only if $n \le 2m-5$ and $(m,n) \notin \{(6,6),(7,7),(8,9)\}$.

With k, a, and b defined as in (1), the conditions $1 \le m \le n$, $(m-d-1)(n-d-1) \le d+1 \le m$, and $n \ge d+2$ are equivalent to

$$b(a+b) \le k+a+2b-1 \Longleftrightarrow (b-1)(a+b-1) \le k,$$

$a \ge 0$, $b \ge 0$, $a+b \ge 1$, and $k+a+3b \ge 2$. Moreover, the conditions $d = \nu_{m,n}$, $m \le n \le 2m-5$, and $(m,n) \notin \{(6,6),(7,7),(8,9)\}$ are equivalent to $k \in \{0,1,2\}$, $a \ge 0$, $b \ge 2$, and $(k,a,b) \notin \{(1,0,2),(2,0,2),(2,1,2)\}$.

2. Four long exact sequences

We present four long exact sequences relating different families of chessboard complexes. In this paper, we will only use the third and the fourth sequences; we list the other two sequences for reference. Throughout this section, we consider an arbitrary coefficient ring $\mathbb{F}$, which we suppress from notation for convenience.

2.1. Long exact sequence relating $\mathsf{M}_{m,n}$, $\mathsf{M}_{m,n-1}$, and $\mathsf{M}_{m-1,n-1}$.

Theorem 2.1. *Define*

$$P_d^{m-1,n-1} = \bigoplus_{s=1}^{m} s\overline{1} \otimes \tilde{H}_d(\mathsf{M}_{[m]\setminus\{s\},[2,n]}).$$

For each $m \ge 1$ and $n \ge 1$, we have a long exact sequence

$$\begin{array}{ccccc} & & \cdots & \longrightarrow & P_d^{m-1,n-1} \\ \longrightarrow & \tilde{H}_d(\mathsf{M}_{m,n-1}) & \longrightarrow & \tilde{H}_d(\mathsf{M}_{m,n}) & \longrightarrow & P_{d-1}^{m-1,n-1} \\ \longrightarrow & \tilde{H}_{d-1}(\mathsf{M}_{m,n-1}) & \longrightarrow & \cdots . \end{array}$$

Proof. This is the long exact sequence for the pair $(\mathsf{M}_{m,n}, \mathsf{M}_{m,n-1})$. □

We refer to this sequence as the *00-01-11 sequence*, thereby indicating that the sequence relates $\mathsf{M}_{m-0,n-0}$, $\mathsf{M}_{m-0,n-1}$, and $\mathsf{M}_{m-1,n-1}$. Note that the sequence is asymmetric in m and n; swapping the indices, we obtain an exact sequence relating $\mathsf{M}_{m,n}$, $\mathsf{M}_{m-1,n}$, and $\mathsf{M}_{m-1,n-1}$.

2.2. Long exact sequence relating $\mathsf{M}_{m,n}$, $\mathsf{M}_{m-1,n-2}$, $\mathsf{M}_{m-2,n-1}$, and $\mathsf{M}_{m-2,n-2}$.

Theorem 2.2 (Shareshian & Wachs [19])**.** *Define*

$$\begin{aligned} P_d^{m-1,n-2} &= \bigoplus_{t=2}^{n} 1\bar{t} \otimes \tilde{H}_d(\mathsf{M}_{[2,m],[2,n]\setminus\{t\}}); \\ Q_d^{m-2,n-1} &= \bigoplus_{s=2}^{m} s\bar{1} \otimes \tilde{H}_d(\mathsf{M}_{[2,m]\setminus\{s\},[2,n]}); \\ R_d^{m-2,n-2} &= \bigoplus_{s=2}^{m} \bigoplus_{t=2}^{n} 1\bar{t} \wedge s\bar{1} \otimes \tilde{H}_d(\mathsf{M}_{[2,m]\setminus\{s\},[2,n]\setminus\{t\}}). \end{aligned}$$

For each $m \geq 2$ and $n \geq 2$, we have a long exact sequence

$$\begin{array}{lcl} & \cdots & \longrightarrow R_{d-1}^{m-2,n-2} \\ \longrightarrow P_{d-1}^{m-1,n-2} \oplus Q_{d-1}^{m-2,n-1} & \longrightarrow \tilde{H}_d(\mathsf{M}_{m,n}) & \longrightarrow R_{d-2}^{m-2,n-2} \\ \longrightarrow P_{d-2}^{m-1,n-2} \oplus Q_{d-2}^{m-2,n-1} & \longrightarrow \quad \cdots . & \end{array}$$

We refer to this sequence as the *00-12-21-22 sequence*. The sequence played an important part in Shareshian and Wachs' analysis [19] of the bottom nonvanishing homology of $\mathsf{M}_{m,n}$. Note that the sequence is symmetric in m and n.

2.3. **Long exact sequence relating $\mathsf{M}_{m,n}$, $\Gamma_{m,n}$, and $\mathsf{M}_{m-1,n-1}$.** The sequence in this section is very similar, but not identical, to the 00-01-11 sequence in Section 2.1. Define

$$\Gamma_{m,n} = \{\sigma \in \mathsf{M}_{m,n} : s\bar{1} \notin \sigma \text{ for } s \in [3,m]\}. \tag{2}$$

Theorem 2.3. *Define*

$$\hat{P}_d^{m-1,n-1} = \bigoplus_{s=3}^{m} s\bar{1} \otimes \tilde{H}_d(\mathsf{M}_{[m]\setminus\{s\},[2,n]});$$

note that this definition differs from that in Section 2.1*. For each $m \geq 1$ and $n \geq 1$, we have a long exact sequence*

$$\begin{array}{lcl} & \cdots & \longrightarrow \hat{P}_d^{m-1,n-1} \\ \longrightarrow \tilde{H}_d(\Gamma_{m,n}) & \longrightarrow \tilde{H}_d(\mathsf{M}_{m,n}) & \longrightarrow \hat{P}_{d-1}^{m-1,n-1} \\ \longrightarrow \tilde{H}_{d-1}(\Gamma_{m,n}) & \longrightarrow \quad \cdots . & \end{array}$$

Proof. This is the long exact sequence for the pair $(\mathsf{M}_{m,n}, \Gamma_{m,n})$. ☐

We refer to this sequence as the *00-Γ-11 sequence*. Note that the sequence is asymmetric in m and n.

2.4. **Long exact sequence relating $\Gamma_{m,n}$, $\mathsf{M}_{m-2,n-1}$, and $\mathsf{M}_{m-2,n-3}$.** Recall the definition of $\Gamma_{m,n}$ from (2).

Theorem 2.4. *Write*

$$\begin{aligned} Q_d^{m-2,n-1} &= (1\overline{1} - 2\overline{1}) \otimes \tilde{H}_d(\mathsf{M}_{[3,m],[2,n]}); \\ R_d^{m-2,n-3} &= \bigoplus_{s \neq t \in [2,n]} 1\overline{s} \wedge 2\overline{t} \otimes \tilde{H}_d(\mathsf{M}_{[3,m],[2,n]\setminus\{s,t\}}). \end{aligned}$$

For each $m \geq 2$ and $n \geq 3$, we have a long exact sequence

$$\begin{array}{cccccc} & & & \cdots & \longrightarrow & R_{d-1}^{m-2,n-3} \\ \xrightarrow{\varphi^*} & Q_{d-1}^{m-2,n-1} & \xrightarrow{\iota^*} & \tilde{H}_d(\Gamma_{m,n}) & \longrightarrow & R_{d-2}^{m-2,n-3} \\ \longrightarrow & Q_{d-2}^{m-2,n-1} & \longrightarrow & \cdots, & & \end{array}$$

where φ^* is induced by the map φ defined by

$$\varphi(1\overline{s} \wedge 2\overline{t} \otimes x) = (1\overline{1} - 2\overline{1}) \otimes x.$$

and ι^* is induced by the natural map $\iota((1\overline{1} - 2\overline{1}) \otimes x) = (1\overline{1} - 2\overline{1}) \wedge x$.

Proof. Define a filtration

$$\Delta_{m,n}^0 \subset \Delta_{m,n}^1 \subset \Delta_{m,n}^2 = \Gamma_{m,n}$$

as follows:

- $\Delta_{m,n}^2 = \Gamma_{m,n}$.
- $\Delta_{m,n}^1$ is the subcomplex of $\Delta_{m,n}^2$ obtained by removing all faces containing $\{1\overline{s}, 2\overline{t}\}$ for some $s, t \in [2, n]$.
- $\Delta_{m,n}^0$ is the subcomplex of $\Delta_{m,n}^1$ obtained by removing the elements $1\overline{2}, \ldots, 1\overline{n}$ and $2\overline{2}, \ldots, 2\overline{n}$.

Writing $\Delta_{m,n}^{-1} = \emptyset$, let us examine $\Delta_{m,n}^i \setminus \Delta_{m,n}^{i-1}$ for $i = 0, 1, 2$.

- $i = 0$. Note that

$$\Delta_{m,n}^0 = \mathsf{M}_{2,1} * \mathsf{M}_{[3,m],[2,n]} \cong \mathsf{M}_{2,1} * \mathsf{M}_{m-2,n-1}.$$

As a consequence,

$$\tilde{H}_d(\Delta_{m,n}^0) \cong (1\overline{1} - 2\overline{1}) \otimes \tilde{H}_{d-1}(\mathsf{M}_{[3,m],[2,n]}) = Q_{d-1}^{m-2,n-1}.$$

- $i = 1$. Observe that

$$\Delta_{m,n}^1 \setminus \Delta_{m,n}^0 = \bigcup_{a=1}^{2} \bigcup_{u=2}^{n} \{\{a\overline{u}\}\} * \mathsf{M}_{\{3-a\},\{1\}} * \mathsf{M}_{[3,m],[2,n]\setminus\{u\}}.$$

It follows that

$$\tilde{H}_d(\Delta_{m,n}^1, \Delta_{m,n}^0) = \bigoplus_{a,u} a\overline{u} \otimes \tilde{H}_{d-1}(\mathsf{M}_{\{3-a\},\{1\}} * \mathsf{M}_{[3,m],[2,n]\setminus\{u\}}) = 0;$$

$\mathsf{M}_{\{3-a\},\{1\}} \cong \mathsf{M}_{1,1}$ is a point. In particular, $\tilde{H}_d(\Delta^1_{m,n}) \cong \tilde{H}_d(\Delta^0_{m,n})$.

- $i = 2$. We have that

$$\Delta^2_{m,n} \setminus \Delta^1_{m,n} = \bigcup_{s,t\in[2,n]} \{\{1\overline{s}, 2\overline{t}\}\} * \mathsf{M}_{[3,m],[2,n]\setminus\{s,t\}};$$

we may hence conclude that

$$\tilde{H}_d(\Delta^2_{m,n}, \Delta^1_{m,n}) = \bigoplus_{s,t} 1\overline{s} \wedge 2\overline{t} \otimes \tilde{H}_{d-2}(\mathsf{M}_{[3,m],[2,n]\setminus\{s,t\}}) = R^{m-2,n-3}_{d-1}.$$

By the long exact sequence for the pair $(\Delta^2_{m,n}, \Delta^1_{m,n})$, it remains to prove that the induced map φ^* has properties as stated in the theorem. Now, in the long exact sequence for $(\Delta^2_{m,n}, \Delta^1_{m,n})$, the induced boundary map from $\tilde{H}_{d+1}(\Delta^2_{m,n}, \Delta^1_{m,n})$ to $\tilde{H}_d(\Delta^1_{m,n})$ maps the element $1\overline{s} \wedge 2\overline{t} \otimes z$ to $(2\overline{t} - 1\overline{s}) \otimes z$. Since

$$(2\overline{t} - 1\overline{s}) \otimes z - \partial((1\overline{1} \wedge 2\overline{t} + 1\overline{s} \wedge 2\overline{1}) \otimes z) = (1\overline{1} - 2\overline{1}) \otimes z,$$

we are done. $\square$

We refer to the sequence in Theorem 2.4 as the *Γ-21-23 sequence*. Note that the sequence is asymmetric in m and n.

3. Bottom nonvanishing homology

Using the long exact sequences in Sections 2.3 and 2.4, we give a computer-free proof that $\tilde{H}_2(\mathsf{M}_{5,5}; \mathbb{Z})$ is a group of size three. While the proof is complicated, our hope is that it may provide at least some insight into the structure of $\mathsf{M}_{5,5}$ and related chessboard complexes.

Theorem 3.1. *We have that $\tilde{H}_2(\mathsf{M}_{5,5}; \mathbb{Z}) \cong \mathbb{Z}_3$.*

Proof. First, we examine $\mathsf{M}_{3,4}$; for alignment with later parts of the proof, we consider $\mathsf{M}_{[3,5],[2,5]}$, thereby shifting the first index two steps and the second index one step. The long exact 00-Γ-11 sequence from Section 2.3 becomes

$$\begin{array}{l} 0 \longrightarrow \tilde{H}_2(\Gamma_{[3,5],[2,5]}) \longrightarrow \tilde{H}_2(\mathsf{M}_{[3,5],[2,5]}) \xrightarrow{\omega^*} 5\overline{2} \otimes \tilde{H}_1(\mathsf{M}_{[3,4],[3,5]}) \\ \quad \longrightarrow \tilde{H}_1(\Gamma_{[3,5],[2,5]}) \xrightarrow{\iota^*} \tilde{H}_1(\mathsf{M}_{[3,5],[2,5]}) \longrightarrow 0. \end{array}$$

As Shareshian and Wachs observed [19, §6], the complex $\mathsf{M}_{m,m+1}$ is an orientable pseudomanifold of dimension $m-1$. In particular, $\mathsf{M}_{[3,5],[2,5]}$ and $\mathsf{M}_{[3,4],[3,5]}$ are orientable pseudomanifolds of dimensions 2 and 1, respectively. Moreover, the top homology group of $\mathsf{M}_{[3,5],[2,5]}$ is generated by

$$z = \sum_{\pi\in\mathfrak{S}_{[2,5]}} \mathrm{sgn}(\pi) \cdot 3\overline{\pi(3)} \wedge 4\overline{\pi(4)} \wedge 5\overline{\pi(5)},$$

and the top homology group of $\mathsf{M}_{[3,4],[3,5]}$ is generated by

$$z' = \sum_{\pi \in \mathfrak{S}_{[3,5]}} \operatorname{sgn}(\pi) \cdot 3\overline{\pi(3)} \wedge 4\overline{\pi(4)}.$$

Since $\omega^*(z) = -z'$, the map ω^* is an isomorphism. As a consequence, the map ι^* induced by the natural inclusion map is also an isomorphism.

The long exact Γ-21-23 sequence for $\Gamma_{[3,5],[2,5]}$ from Section 2.4 becomes

$$0 \longrightarrow (3\overline{2} - 4\overline{2}) \otimes \tilde{H}_0(\mathsf{M}_{\{5\},[3,5]}) \xrightarrow{\iota^*} \tilde{H}_1(\Gamma_{[3,5],[2,5]}) \longrightarrow 0,$$

which yields that each of $\tilde{H}_1(\Gamma_{[3,5],[2,5]})$ and $\tilde{H}_1(\mathsf{M}_{[3,5],[2,5]})$ is generated by $e_i = (3\overline{2} - 4\overline{2}) \wedge (5\overline{3} - 5\overline{i})$ for $i \in \{4, 5\}$.

Now, consider $\mathsf{M}_{5,5}$. The tail end of the Γ-21-23 sequence is

$$\begin{aligned} &\bigoplus_{s,t} 1\overline{s} \wedge 2\overline{t} \otimes \tilde{H}_1(\mathsf{M}_{[3,5],[2,5]\setminus\{s,t\}}) \\ \xrightarrow{\varphi^*} \quad &(1\overline{1} - 2\overline{1}) \otimes \tilde{H}_1(\mathsf{M}_{[3,5],[2,5]}) \quad \xrightarrow{\iota^*} \tilde{H}_2(\Gamma_{5,5}) \to 0, \end{aligned}$$

where the first sum ranges over all pairs of distinct elements $s, t \in [2, 5]$. Writing $\{s, t, u, v\} = [2, 5]$, we note that $\tilde{H}_1(\mathsf{M}_{[3,5],[2,5]\setminus\{s,t\}}) = \tilde{H}_1(\mathsf{M}_{[3,5],\{u,v\}})$ is generated by the cycle

$$z_{uv} = 3\overline{u} \wedge 4\overline{v} + 4\overline{v} \wedge 5\overline{u} + 5\overline{u} \wedge 3\overline{v} + 3\overline{v} \wedge 4\overline{u} + 4\overline{u} \wedge 5\overline{v} + 5\overline{v} \wedge 3\overline{u}.$$

By Theorem 2.4, φ^* maps $1\overline{s} \wedge 2\overline{t} \otimes z_{uv}$ to $(1\overline{1} - 2\overline{1}) \otimes z_{uv}$. Since $z_{uv} = z_{vu}$, we conclude that the image under φ^* is generated by the six cycles $z_{23}, z_{24}, z_{25}, z_{34}, z_{35}, z_{45}$.

In $\tilde{H}_1(\mathsf{M}_{[3,5],[2,5]})$, we have that $z_{st} = z_{uv}$, because $z_{st} - z_{uv}$ equals the boundary of

$$\begin{aligned} \gamma \;=\; & 3\overline{u} \wedge 5\overline{s} \wedge 4\overline{v} - 5\overline{s} \wedge 4\overline{v} \wedge 3\overline{t} + 4\overline{v} \wedge 3\overline{t} \wedge 5\overline{u} - 3\overline{t} \wedge 5\overline{u} \wedge 4\overline{s} \\ + \; & 5\overline{u} \wedge 4\overline{s} \wedge 3\overline{v} - 4\overline{s} \wedge 3\overline{v} \wedge 5\overline{t} + 3\overline{v} \wedge 5\overline{t} \wedge 4\overline{u} - 5\overline{t} \wedge 4\overline{u} \wedge 3\overline{s} \\ + \; & 4\overline{u} \wedge 3\overline{s} \wedge 5\overline{v} - 3\overline{s} \wedge 5\overline{v} \wedge 4\overline{t} + 5\overline{v} \wedge 4\overline{t} \wedge 3\overline{u} - 4\overline{t} \wedge 3\overline{u} \wedge 5\overline{s}. \end{aligned}$$

Namely, γ is of the form $a_1 \wedge a_2 \wedge a_3 - a_2 \wedge a_3 \wedge a_4 + \cdots - a_{12} \wedge a_1 \wedge a_2$, which yields the boundary $-a_1 \wedge a_3 + a_2 \wedge a_4 - \cdots + a_{12} \wedge a_2$. As a consequence, the image under φ^* is generated by the three cycles z_{34}, z_{35}, z_{45}.

Assume that $s = 2$ and $\{t, u, v\} = \{3, 4, 5\}$ and write

$$\begin{aligned} w_{uv} \;=\; & 5\overline{u} \wedge 4\overline{s} \wedge 3\overline{v} - 4\overline{s} \wedge 3\overline{v} \wedge 5\overline{t} + 3\overline{v} \wedge 5\overline{t} \wedge 4\overline{u} \\ - \; & 5\overline{t} \wedge 4\overline{u} \wedge 3\overline{s} + 4\overline{u} \wedge 3\overline{s} \wedge 5\overline{v}. \end{aligned}$$

We obtain that

$$\begin{aligned}
\partial(w_{uv}+w_{vu}) &= (5\overline{u}\wedge 4\overline{s} - 5\overline{u}\wedge 3\overline{v} + 4\overline{s}\wedge 5\overline{t} - 3\overline{v}\wedge 4\overline{u} + 5\overline{t}\wedge 3\overline{s} \\
&- 4\overline{u}\wedge 5\overline{v} + 3\overline{s}\wedge 5\overline{v}) + (5\overline{v}\wedge 4\overline{s} - 5\overline{v}\wedge 3\overline{u} + 4\overline{s}\wedge 5\overline{t} \\
&- 3\overline{u}\wedge 4\overline{v} + 5\overline{t}\wedge 3\overline{s} - 4\overline{v}\wedge 5\overline{u} + 3\overline{s}\wedge 5\overline{u}) \\
&= (4\overline{s} - 3\overline{s})\wedge(2\cdot 5\overline{t} - 5\overline{u} - 5\overline{v}) - z_{uv}.
\end{aligned}$$

Since $s=2$, it follows that z_{uv} is equal to either $-e_4-e_5$, $2e_4-e_5$, or $-e_4+2e_5$ in $\tilde{H}_1(\mathsf{M}_{[3,5][2,5]})$ depending on the values of t, u, and v.

We conclude that the set $\{\varphi^*(1\overline{s}\wedge 2\overline{t}\otimes z_{uv}) : \{s,t,u,v\} = [2,5]\}$ generates the subgroup $\{(1\overline{1}-2\overline{1})\otimes(ae_4+be_5) : a-b\equiv 0 \pmod 3\}$ of $(1\overline{1}-2\overline{1})\otimes\tilde{H}_1(\mathsf{M}_{[3,5],[2,5]})$. As a consequence, $\tilde{H}_2(\Gamma_{5,5})\cong\mathbb{Z}_3$, and

$$\rho = (1\overline{1}-2\overline{1})\wedge(3\overline{2}-4\overline{2})\wedge(5\overline{3}-5\overline{4})$$

is a generator for this group. Swapping $\overline{3}$ and $\overline{4}$, we obtain $-\rho$; we obtain the same result if we swap 3 and 4 or if we swap 1 and 2. Hence, by symmetry, the group

$$T = \mathfrak{S}_{\{1,2\}}\times\mathfrak{S}_{\{3,4,5\}}\times\mathfrak{S}_{\{\overline{2},\overline{3},\overline{4},\overline{5}\}}$$

acts on $\tilde{H}_2(\Gamma_{5,5})\cong\mathbb{Z}_3$ by $\pi(\rho)=\mathrm{sgn}(\pi)\cdot\rho$.

It remains to prove that $\tilde{H}_2(\Gamma_{5,5})\cong\tilde{H}_2(\mathsf{M}_{5,5})$. For this, consider the tail end of the 00-Γ-11 sequence from Section 2.3:

$$\bigoplus_{x=3}^{5} x\overline{1}\otimes\tilde{H}_2(\mathsf{M}_{[5]\setminus\{x\},[2,5]}) \xrightarrow{\psi^*} \tilde{H}_2(\Gamma_{5,5}) \longrightarrow \tilde{H}_2(\mathsf{M}_{5,5}) \to 0$$

By a result due to Shareshian and Wachs [19, Lemma 5.9], we have that $\tilde{H}_2(\mathsf{M}_{[5]\setminus\{x\},[2,5]})\cong\tilde{H}_2(\mathsf{M}_{4,4})$ is generated by cycles of type $\left[{3,2 \atop 2}\right]\wedge\left[{1,2 \atop 1}\right]$ and cycles of type $\left[{2,3 \atop 2}\right]\wedge\left[{2,1 \atop 1}\right]$; recall notation from Section 1.1. By properties of ψ^*, we need only prove that any such cycle vanishes in $\tilde{H}_2(\Gamma_{5,5})$ whenever $x\in[3,5]$.

- A cycle of the first type is of the form $z=\lambda\cdot\gamma\wedge(d\overline{u}-d\overline{v})$, where λ is a constant scalar,

$$\gamma = a\overline{s}\wedge b\overline{t} + b\overline{t}\wedge c\overline{s} + c\overline{s}\wedge a\overline{t} + a\overline{t}\wedge b\overline{s} + b\overline{s}\wedge c\overline{t} + c\overline{t}\wedge a\overline{s},$$

$\{a,b,c,d\}=[5]\setminus\{x\}$, and $\{s,t,u,v\}=[2,5]$. By the above discussion, swapping $\overline{s}$ and $\overline{t}$ in z should yield $-z$, but obviously the same swap in γ again yields γ, which implies that $z=-z$; hence $z=0$.

- A cycle of the second type is of the form $z=\lambda\cdot\gamma\wedge(c\overline{v}-d\overline{v})$, where λ is a constant scalar, say $\lambda=1$, and

$$\gamma = a\overline{s}\wedge b\overline{t} + b\overline{t}\wedge a\overline{u} + a\overline{u}\wedge b\overline{s} + b\overline{s}\wedge a\overline{t} + a\overline{t}\wedge b\overline{u} + b\overline{u}\wedge a\overline{s};$$

again $\{a,b,c,d\} = [5] \setminus \{x\}$ and $\{s,t,u,v\} = [2,5]$. If $\{a,b\} \subset [3,5]$, then we may swap a and b and again conclude that $z = -z$; the same argument applies if $\{a,b\} = \{1,2\}$. For the remaining case, we may assume that $c \in [1,2]$ and $d \in [3,5]$. Swapping d and x yields $-z = \gamma \wedge (c\overline{v} - x\overline{v})$; recall that $x \in [3,5]$. As a consequence,

$$2z = z - (-z) = \gamma \wedge (x\overline{v} - d\overline{v}) = \partial(c\overline{1} \wedge \gamma \wedge (x\overline{v} - d\overline{v}));$$

hence z is again zero. Namely, since $c \in [1,2]$, we have that $c\overline{1}$ is an element in $\Gamma_{5,5}$. As a consequence, ψ^* is the zero map as desired. $\square$

By Theorems 1.1 and 1.2, the connectivity degree of $\mathsf{M}_{m,n}$ is exactly $\nu_{m,n} - 1$ whenever $n \geq 2m-4$ or $(m,n) \in \{(6,6),(7,7),(8,9)\}$. As mentioned in the introduction, Shareshian and Wachs [19] extended this result to all $(m,n) \neq (1,1)$, thereby settling a conjecture due to Björner et al. [4]:

Theorem 3.2 (Shareshian & Wachs [19]). *If $m \leq n \leq 2m-5$ and $(m,n) \neq (8,9)$, then there is nonvanishing 3-torsion in $\tilde{H}_{\nu_{m,n}}(\mathsf{M}_{m,n};\mathbb{Z})$. If in addition $m+n \equiv 1 \pmod 3$, then $\tilde{H}_{\nu_{m,n}}(\mathsf{M}_{m,n};\mathbb{Z}) \cong \mathbb{Z}_3$.*

By Theorem 4.4 in Section 4.1, there is nonvanshing 3-torsion also in $\tilde{H}_{\nu_{8,9}}(\mathsf{M}_{8,9};\mathbb{Z})$; in that theorem, choose $(k,a,b) = (2,1,2)$.

[Table 1]

In fact, Shareshian and Wachs provided much more specific information about the exponent of $\tilde{H}_{\nu_{m,n}}(\mathsf{M}_{m,n};\mathbb{Z})$; see Table 1.

Conjecture 3.3 (Shareshian & Wachs [19]). *The group $\tilde{H}_{\nu_{m,n}}(\mathsf{M}_{m,n};\mathbb{Z})$ is torsion-free if and only if $n \geq 2m-4$.*

The conjecture is known to be true in all cases but $n = 2m-4$ and $n = 2m-3$; Shareshian and Wachs [19] settled the case $n = 2m-2$.

Corollary 3.4 (Shareshian & Wachs [19]). *For all $(m,n) \neq (1,1)$, we have that $\tilde{H}_{\nu_{m,n}}(\mathsf{M}_{m,n};\mathbb{Z})$ is nonzero.*

4. Higher-degree homology

In Section 4.1, we detect 3-torsion in higher-degree homology groups of $\mathsf{M}_{m,n}$. In Section 4.2, we proceed with upper bounds on the dimension of the homology over $\mathbb{Z}_3$.

4.1. 3**-torsion in higher-degree homology groups.** This section builds on work previously published in the author's thesis [10, 11]. Fix $n, d \geq 0$ and let γ be an element in $\tilde{H}_{d-1}(\mathsf{M}_n; \mathbb{Z})$; note that we consider the matching complex M_n. For each $k \geq 0$, define a map

$$\begin{cases} \theta_k : \tilde{H}_{k-1}(\mathsf{M}_{k,k+1}; \mathbb{Z}) \to \tilde{H}_{k-1+d}(\mathsf{M}_{2k+1+n}; \mathbb{Z}) \\ \theta_k(z) = z \wedge \gamma^{(2k+1)}, \end{cases}$$

where we obtain $\gamma^{(2k+1)}$ from γ by replacing each occurrence of the vertex i with $i + 2k + 1$ for every $i \in [n]$.

For any prime p, we have that θ_k induces a homomorphism

$$\theta_k \otimes_{\mathbb{Z}} \iota_p : \tilde{H}_{k-1}(\mathsf{M}_{k,k+1}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \tilde{H}_{k-1+d}(\mathsf{M}_{2k+1+n}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p,$$

where $\iota_p : \mathbb{Z}_p \to \mathbb{Z}_p$ is the identity. The following result about the matching complex is a special case of a more general result from a previous paper [12].

Theorem 4.1 (Jonsson [12])**.** *Fix $k_0 \geq 0$. With notation and assumptions as above, if $\theta_{k_0} \otimes_{\mathbb{Z}} \iota_p$ is a monomorphism, then $\theta_k \otimes_{\mathbb{Z}} \iota_p$ is a monomorphism for each $k \geq k_0$.*

As alluded to in the proof of Theorem 3.1 in Section 3, we have that $\mathsf{M}_{k,k+1}$ is an orientable pseudomanifold of dimension $k - 1$; hence $\tilde{H}_{k-1}(\mathsf{M}_{k,k+1}; \mathbb{Z}) \cong \mathbb{Z}$. Shareshian and Wachs [19, §6] observed that this group is generated by the cycle

$$z_{k,k+1} = \sum_{\pi \in \mathfrak{S}_{[k+1]}} \mathrm{sgn}(\pi) \cdot 1\overline{\pi(1)} \wedge \cdots \wedge k\overline{\pi(k)}.$$

Note that the sum is over all permutations on $k + 1$ elements. Theorem 4.1 implies the following result.

Corollary 4.2. *With notation and assumptions as in Theorem* 4.1*, if $(z_{k_0,k_0+1} \wedge \gamma^{(2k_0+1)}) \otimes 1$ is nonzero in $\tilde{H}_{k_0-1+d}(\mathsf{M}_{2k_0+1+n}; \mathbb{Z}) \otimes \mathbb{Z}_p$, then $(z_{k,k+1} \wedge \gamma^{(2k+1)}) \otimes 1$ is nonzero in $\tilde{H}_{k-1+d}(\mathsf{M}_{2k+1+n}; \mathbb{Z}) \otimes \mathbb{Z}_p$ for all $k \geq k_0$.*

We will also need a result about the bottom nonvanishing homology of the matching complex. Define

$$\begin{aligned} \gamma_{3r} &= (12 - 23) \wedge (45 - 56) \wedge (78 - 89) \\ &\quad \wedge \cdots \wedge ((3r-2)(3r-1) - (3r-1)(3r)); \end{aligned} \tag{3}$$

this is a cycle in both $\tilde{C}_{r-1}(\mathsf{M}_{3r}; \mathbb{Z})$ and $\tilde{C}_{r-1}(\mathsf{M}_{3r+1}; \mathbb{Z})$.

Theorem 4.3 (Bouc [5])**.** *For $r \geq 2$, we have that $\tilde{H}_{r-1}(\mathsf{M}_{3r+1}; \mathbb{Z}) \cong \mathbb{Z}_3$. Moreover, this group is generated by γ_{3r} and hence by any element obtained from γ_{3r} by permuting the underlying vertex set.*

Assume that $m + n \equiv 0 \pmod 3$ and $m \le n \le 2m$. Define the cycle $\gamma_{m,n}$ in $\tilde{H}_{\nu_{m,n}}(\mathsf{M}_{m,n};\mathbb{Z})$ recursively as follows, the base case being $\gamma_{1,2} = 1\overline{1} - 1\overline{2}$:

$$\gamma_{m,n} = \begin{cases} \gamma_{m-1,n-2} \wedge (m(\overline{n-1}) - m\overline{n}) & \text{if } m < n; \\ \gamma_{m-2,n-1} \wedge ((m-1)\overline{n} - m\overline{n}) & \text{if } m = n. \end{cases} \tag{4}$$

For $n > m$, we define $\gamma_{n,m}$ by replacing $i\overline{j}$ with $j\overline{i}$ in $\gamma_{m,n}$ for each $i \in [m]$ and $j \in [n]$.

Recall that $\nu_{m,n} = \frac{m+n-4}{3}$ whenever $m \le n \le 2m-2$.

Theorem 4.4. *There is* 3*-torsion in* $\tilde{H}_d(\mathsf{M}_{m,n};\mathbb{Z})$ *whenever*

$$\begin{cases} m+1 \le n \le 2m-5 \\ \left\lceil \frac{m+n-4}{3} \right\rceil \le d \le m-3 \end{cases} \iff \begin{cases} k & \ge & 0 \\ a & \ge & 1 \\ b & \ge & 2, \end{cases}$$

where k*,* a*, and* b *are defined as in* (1)*. Moreover, there is* 3*-torsion in* $\tilde{H}_d(\mathsf{M}_{m,m};\mathbb{Z})$ *whenever*

$$\left\lceil \frac{2m-4}{3} \right\rceil \le d \le m-4 \iff \begin{cases} k & \ge & 0 \\ a & = & 0 \\ b & \ge & 3. \end{cases}$$

Proof. Assume that $k \ge 0$, $a \ge 1$, and $b \ge 2$. Writing $m_0 = a + 3b - 2$ and $n_0 = 2a + 3b - 3$, we have the inequalities

$$a + 3b - 2 \le 2a + 3b - 3 \le 2a + 6b - 9 \iff m_0 \le n_0 \le 2m_0 - 5. \tag{5}$$

Note that $m_0 + n_0 = 3a + 6b - 5 \equiv 1 \pmod 3$. Define

$$w_{k+1} = z_{k+1,k+2} \wedge \gamma^{(k+1,k+2)}_{m_0,n_0-1},$$

where we obtain $\gamma^{(k+1,k+2)}_{m_0,n_0-1}$ from the cycle γ_{m_0,n_0-1} defined in (4) by replacing $i\overline{j}$ with $(i+k+1)(\overline{j+k+2})$. View γ_{m_0,n_0-1} as an element in the homology of M_{m_0,n_0}. Since $z_{k+1,k+2}$ has type $\left[\begin{smallmatrix} k+1,k+2 \\ k+1 \end{smallmatrix}\right]$ and since γ_{m_0,n_0-1} has type $\left[\begin{smallmatrix} a+3b-2,2a+3b-3 \\ a+2b-2 \end{smallmatrix}\right]$ (or rather $\left[\begin{smallmatrix} a+3b-2,2a+3b-4 \\ a+2b-2 \end{smallmatrix}\right] \wedge \left[\begin{smallmatrix} 0,1 \\ 0 \end{smallmatrix}\right]$), we obtain that w_{k+1} has type

$$\begin{bmatrix} k+1+a+3b-2, k+2+2a+3b-3 \\ k+1+a+2b-2 \end{bmatrix} = \begin{bmatrix} m,n \\ d+1 \end{bmatrix};$$

hence we may view w_{k+1} as an element in $\tilde{H}_d(\mathsf{M}_{m,n};\mathbb{Z})$.

Choosing $k = 0$, we obtain that

$$w_1 = z_{1,2} \wedge \gamma^{(1,2)}_{m_0,n_0-1}.$$

We claim that w_1 has order three when viewed as an element in

$$\tilde{H}_{\frac{m_0+n_0-1}{3}}(\mathsf{M}_{m_0+n_0+3};\mathbb{Z}) = \tilde{H}_{a+2b-2}(\mathsf{M}_{3a+6b-2};\mathbb{Z}).$$

Namely, we may relabel the vertices to transform w_1 into the cycle $\gamma_{m_0+n_0+2}$ defined in (3). Since $m_0 + n_0 + 3 \geq 13$, Theorem 4.3 yields the claim.

Applying Corollary 4.2, we conclude that $w_{k+1} \otimes 1$ is a nonzero element in the group $\tilde{H}_{k+a+2b-2}(\mathsf{M}_{2k+3a+6b-2};\mathbb{Z})\otimes\mathbb{Z}_3 = \tilde{H}_d(\mathsf{M}_{m+n};\mathbb{Z})\otimes\mathbb{Z}_3$ for every $k \geq 0$. As a consequence, $w_{k+1} \otimes 1$ is nonzero also in

$$\tilde{H}_{k+a+2b-2}(\mathsf{M}_{k+a+3b-1,k+2a+3b-1};\mathbb{Z}) \otimes \mathbb{Z}_3 = \tilde{H}_d(\mathsf{M}_{m,n};\mathbb{Z}) \otimes \mathbb{Z}_3$$

for every $k \geq 1$. Since $\tilde{H}_{a+b-3}(\mathsf{M}_{m_0,n_0};\mathbb{Z})$ is an elementary 3-group by Theorem 3.2 and (5), the order of γ_{m_0,n_0-1} in $\tilde{H}_r(\mathsf{M}_{m_0,n_0};\mathbb{Z})$ is three. It follows that the order of w_{k+1} in $\tilde{H}_d(\mathsf{M}_{m,n};\mathbb{Z})$ is three as well.

The remaining case is $m = n$, in which case the upper bound on d is $m-4$ rather than $m-3$. Since $a = 0$, we get

$$\left\{\begin{array}{ccccccc} k & = & -2m & + & 3d & + & 4 \\ b & = & m & - & d & - & 1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{ccccccc} m & = & k & + & 3b & - & 1 \\ d & = & k & + & 2b & - & 2. \end{array}\right.$$

Clearly, $k \geq 0$ and $b \geq 3$.

Consider the cycle $w_{k+1} = z_{k+1,k+2} \wedge \gamma^{(k+1,k+2)}_{3b-2,3b-4}$. By Corollary 4.2, $w_{k+1} \otimes 1$ is nonzero in $\tilde{H}_{k+2b-2}(\mathsf{M}_{2k+6b-2};\mathbb{Z}) \otimes \mathbb{Z}_3$. Namely, up to the names of the vertices, w_1 coincides with γ_{6b-3} in (3), which is a nonzero element of order three in the group $\tilde{H}_{2b-2}(\mathsf{M}_{6b-2};\mathbb{Z})$ by Theorem 4.3; $b \geq 3$. We conclude that $w_{k+1} \otimes 1$ is a nonzero element in $\tilde{H}_{k+2b-2}(\mathsf{M}_{k+3b-1,k+3b-1};\mathbb{Z})\otimes\mathbb{Z}_3 = \tilde{H}_d(\mathsf{M}_{m,m};\mathbb{Z})\otimes\mathbb{Z}_3$. Since $3b-3 \geq 6$, we have that $\gamma_{3b-2,3b-4}$ must have order three in $\tilde{H}_{2b-3}(\mathsf{M}_{3b-2,3b-3};\mathbb{Z})$; apply Theorem 3.2. This implies that the same must be true for w_{k+1} in $\tilde{H}_d(\mathsf{M}_{m,m};\mathbb{Z})$. $\square$

Corollary 4.5. *The group $\tilde{H}_5(\mathsf{M}_{8,9};\mathbb{Z}) = \tilde{H}_{\nu_{8,9}}(\mathsf{M}_{8,9};\mathbb{Z})$ contains nonvanishing 3-torsion. As a consequence, there is nonvanishing 3-torsion in $\tilde{H}_{\nu_{m,n}}(\mathsf{M}_{m,n};\mathbb{Z})$ whenever $m \leq n \leq 2m-5$.*

Proof. The first statement is a consequence, of Theorem 4.4; choose $k = 2$, $a = 1$, and $b = 2$. For the second statement, apply Theorem 3.2. $\square$

Theorem 4.6. *For $1 \leq m \leq n$, the group $\tilde{H}_d(\mathsf{M}_{m,n};\mathbb{Z})$ is nonzero if and only if either*

$$\left\lceil \frac{m+n-4}{3} \right\rceil \leq d \leq m-2 \Longleftrightarrow \left\{\begin{array}{ccc} k & \geq & 0 \\ a & \geq & 0 \\ b & \geq & 1 \end{array}\right.$$

or

$$\begin{cases} m & \geq & 1 \\ n & \geq & m+1 \\ d & = & m-1 \end{cases} \Longleftrightarrow \begin{cases} k & \geq & 2-a \\ a & \geq & 1 \\ b & = & 0, \end{cases}$$

where k, a, and b are defined as in (1)*.*

Proof. For homology to exist, we certainly must have that $b \geq 0$, and we restrict to $a \geq 0$ by assumption. Moreover, $b = 0$ means that $d = m - 1$, in which case there is homology only if $m \leq n - 1$, hence $a \geq 1$ and $k + a \geq 2$; for the latter inequality, recall that we restrict our attention to $m \geq 1$. Finally, $k < 0$ reduces to the case $b = 0$, because we then have homology only if $n \geq 2m + 2$ and $d = m - 1$; apply Theorem 1.1.

For the other direction, Theorem 4.4 yields that we only need to consider the following cases:

- $k \geq 0$, $a = 0$, and $b = 2$. By Theorem 1.2, we have infinite homology for $a = 0$ and $b = 2$ if and only if $k \geq (b-1)(a+b-1) = a + 1 = 1$. The remaining case is $(k, a, b) = (0, 0, 2) \Longleftrightarrow (m, n, d) = (5, 5, 2)$, in which case we have nonzero homology by Theorem 3.1.
- $k \geq 0$, $a \geq 0$, and $b = 1$. This time, Theorem 1.2 yields infinite homology for $a \geq 0$ and $b = 1$ as soon as $k \geq 0$.
- $k \geq 2 - a$, $a \geq 1$, and $b = 0$. By yet another application of Theorem 1.2, we have infinite homology for $b = 0$ whenever $a \geq 1$, $k \geq 1 - a$, and $k + a \geq 2$. Since the third inequality implies the second, we are done. $\square$

Conjecture 4.7 (Shareshian & Wachs [19])**.** *For $1 \leq m \leq n$, the group $\tilde{H}_d(\mathsf{M}_{m,n}; \mathbb{Z})$ contains 3-torsion if and only if*

$$\begin{cases} m \leq n \leq 2m-5 \\ \\ \left\lceil \frac{m+n-4}{3} \right\rceil \leq d \leq m-3 \end{cases} \Longleftrightarrow \begin{cases} k & \geq & 0 \\ a & \geq & 0 \\ b & \geq & 2. \end{cases}$$

Note that Conjecture 4.7 implies Conjecture 3.3. Conjecture 4.7 remains unsettled in the following cases:

- $d = m - 2$: $9 \leq m + 2 \leq n \leq 2m - 3$. Equivalently, $k \geq 1$, $a \geq 2$, and $b = 1$. Conjecture: There is no 3-torsion.
- $d = m - 3$: $8 \leq m = n$. Equivalently, $k \geq 3$, $a = 0$, and $b = 2$. Conjecture: There is 3-torsion.

The conjecture is fully settled for $n = m + 1$ and $n \geq 2m - 2$; see Shareshian and Wachs [19] for the case $n = 2m - 2$, and use Theorem 1.1 for the case $n \geq 2m - 1$. For the case $n = m + 1$, we have

that $\tilde{H}_{m-2}(\mathsf{M}_{m,m+1};\mathbb{Z})$ is torsion-free, because $\mathsf{M}_{m,m+1}$ is an orientable pseudomanifold; see Spanier [20, Ex. 4.E.2].

4.2. **Bounds on the homology over $\mathbb{Z}_3$.** Fix a field $\mathbb{F}$ and let

$$\begin{aligned}\beta_d^{m,n} &= \dim_{\mathbb{F}} \tilde{H}_d(\mathsf{M}_{m,n};\mathbb{F});\\ \alpha_d^{m,n} &= \dim_{\mathbb{F}} \tilde{H}_d(\Gamma_{m,n};\mathbb{F});\end{aligned}$$

$\Gamma_{m,n}$ is defined as in (2).

Lemma 4.8. *For each $m \ge 2$ and $n \ge 3$, we have that*

$$\beta_d^{m,n} \le \beta_{d-1}^{m-2,n-1} + (m-2)\beta_{d-1}^{m-1,n-1} + 2\binom{n-1}{2}\beta_{d-2}^{m-2,n-3}.$$

Thus, by symmetry,

$$\beta_d^{m,n} \le \beta_{d-1}^{m-1,n-2} + (n-2)\beta_{d-1}^{m-1,n-1} + 2\binom{m-1}{2}\beta_{d-2}^{m-3,n-2}$$

whenever $m \ge 3$ and $n \ge 2$.

Proof. By the long exact 00-Γ-11 sequence in Section 2.3, we have that

$$\beta_d^{m,n} \le \alpha_d^{m,n} + (m-2)\beta_{d-1}^{m-1,n-1}.$$

Moreover, the long exact Γ-21-23 sequence in Section 2.4 yields the inequality

$$\alpha_d^{m,n} \le \beta_{d-1}^{m-2,n-1} + 2\binom{n-1}{2}\beta_{d-2}^{m-2,n-3}.$$

Summing, we obtain the desired inequality. □

Define $\hat{\beta}_k^{a,b} = \beta_d^{m,n}$, where k, a, and b are defined as in (1). We may rewrite the second inequality in Lemma 4.8 as follows:

Corollary 4.9. *We have that*

$$\hat{\beta}_k^{a,b} \le \hat{\beta}_k^{a-1,b} + (k+2a+3b-3)\hat{\beta}_{k-1}^{a,b} + 2\binom{k+a+3b-2}{2}\hat{\beta}_{k-1}^{a+1,b-1} \tag{6}$$

for $k \ge 0$, $a \ge 0$, and $b \ge 2$.

Theorem 4.10. *With $\mathbb{F} = \mathbb{Z}_3$ and $d = \nu_{m,n}$, the second bound in Lemma 4.8 is sharp whenever $m \le n \le 2m-5$, $m+n \equiv 1 \pmod 3$, and $(m,n) \ne (5,5)$. Equivalently, the bound is sharp whenever $k=0$, $a \ge 0$, $b \ge 2$, and $(k,a,b) \ne (0,0,2)$, where k, a, and b are defined as in (1).*

Proof. Since $\hat{\beta}_0^{a,b} = 1$ for $a \ge 0$ and $b \ge 2$ by Theorem 3.2, it suffices to prove that

$$\hat{\beta}_0^{a-1,b} + (2a+3b-3)\hat{\beta}_{-1}^{a,b} + 2\binom{a+3b-2}{2}\hat{\beta}_{-1}^{a+1,b-1} = 1 \tag{7}$$

for all a and b as in the theorem; apply Corollary 4.9. Clearly, $\hat{\beta}_0^{a-1,b} = 1$; when $a = 0$, use the fact that $\hat{\beta}_0^{-1,b} = \hat{\beta}_0^{1,b-1}$. Moreover, Theorem 1.1

yields that $\hat{\beta}_{-1}^{a,b} = 0$ whenever $a \geq 0$ and $b \geq 1$. As a consequence, we are done. $\square$

Theorem 4.11. *For each $k \geq 0$, there is a polynomial $f_k(a,b)$ of degree $3k$ such that $\hat{\beta}_k^{a,b} \leq f_k(a,b)$ whenever $a \geq 0$ and $b \geq k+2$ and such that*

$$f_k(a,b) = \frac{1}{3^k k!}\left((a+3b)^3 - 9b^3\right)^k + \epsilon_k(a,b)$$

for some polynomial $\epsilon_k(a,b)$ of degree at most $3k-1$. Equivalently,

$$\beta_d^{m,n} \leq f_{3d-m-n+4}(n-m, m-d-1)$$

for $m \leq n \leq 2m-5$ and $\frac{m+n-4}{3} \leq d \leq \frac{2m+n-7}{4}$.

Proof. The case $k = 0$ is a consequence of Theorem 3.2. Assume that $k \geq 1$ and $b > k+2$.

First, assume that $a > 0$. Induction and Corollary 4.9 yield that

$$\begin{aligned} \hat{\beta}_k^{a,b} - \hat{\beta}_k^{a-1,b} &\leq (k+2a+3b-3) f_{k-1}(a,b) \\ &+ 2\tbinom{k+a+3b-2}{2} f_{k-1}(a+1,b-1), \end{aligned}$$

where f_{k-1} is a polynomial with properties as in the theorem. The right-hand side is of the form

$$g_k(a,b) = \frac{1}{3^{k-1}(k-1)!}\left((a+3b)^3 - 9b^3\right)^{k-1}(a+3b)^2 + h_k(a,b),$$

where $h_k(a,b)$ is a polynomial of degree at most $3k-2$. Now,

$$\begin{aligned} &\frac{1}{3^{k-1}(k-1)!}\left((a+3b)^3 - 9b^3\right)^{k-1}(a+3b)^2 \\ = \; &\frac{1}{3^{k-1}(k-1)!}\sum_{\ell=0}^{k-1}\binom{k-1}{l}(a+3b)^{3k-3\ell-1}(-9b^3)^\ell. \end{aligned}$$

Summing over a, we obtain that

$$\hat{\beta}_k^{a,b} \leq \hat{\beta}_k^{0,b} + \sum_{i=1}^{a} g_k(i,b).$$

The right-hand side is a polynomial in a and b with dominating term

$$\begin{aligned} &\frac{1}{3^{k-1}(k-1)!}\sum_{\ell=0}^{k-1}\binom{k-1}{l}\frac{(a+3b)^{3k-3\ell} - (3b)^{3k-3\ell}}{3k-3\ell}(-9b^3)^\ell \\ = \; &\frac{1}{3^k k!}\sum_{\ell=0}^{k}\binom{k}{l}\left(((a+3b)^3)^{k-\ell} - (27b^3)^{k-\ell}\right)(-9b^3)^\ell \\ = \; &\frac{1}{3^k k!}\left((a+3b)^3 - 9b^3\right)^k - \frac{1}{3^k k!}(18b^3)^k. \end{aligned} \tag{8}$$

Proceeding with $\hat{\beta}_k^{0,b}$ for $b \geq k+3$, note that $\hat{\beta}_k^{-1,b} = \hat{\beta}_k^{1,b-1}$. As a consequence,

$$\begin{aligned} \hat{\beta}_k^{0,b} &\leq \hat{\beta}_k^{1,b-1} + (k+3b-3)\hat{\beta}_{k-1}^{0,b} + 2\tbinom{k+3b-2}{2}\hat{\beta}_{k-1}^{1,b-1} \\ &\leq \hat{\beta}_k^{0,b-1} + (k+3b-4)\hat{\beta}_{k-1}^{1,b-1} + 2\tbinom{k+3b-4}{2}\hat{\beta}_{k-1}^{2,b-2} \\ &+ (k+3b-3)\hat{\beta}_{k-1}^{0,b} + 2\tbinom{k+3b-2}{2}\hat{\beta}_{k-1}^{1,b-1}. \end{aligned}$$

Using induction, we conclude that

$$\begin{aligned} \hat{\beta}_k^{0,b} &\leq \hat{\beta}_k^{0,b-1} + 9b^2 f_{k-1}(2,b-2) + 9b^2 f_{k-1}(1,b-1) + O(b^{3k-2}) \\ &= 18b^2 \frac{(18b^3)^{k-1}}{3^{k-1}(k-1)!} + O(b^{3k-2}) = \frac{18^k b^{3k-1}}{3^{k-1}(k-1)!} + O(b^{3k-2}), \end{aligned}$$

where f_{k-1} is a polynomial with properties as in the theorem. Summing over b, we may conclude that $\hat{\beta}_k^{0,b}$ is bounded by a polynomial in b with dominating term $\frac{18^k b^{3k}}{3^k k!}$. Combined with (8), this yields the theorem. $\square$

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TABLE 1. The exponent $\epsilon_{m,n}$ of $\tilde{H}_{\nu_{m,n}}(\mathsf{M}_{m,n};\mathbb{Z})$ for $m \le n \le 2m-5$ and $(m,n) \notin \{(6,6),(7,7),(8,9)\}$. On the right we give the values k, a, and b defined as in (1).

$2m-n$	Restriction	$\epsilon_{m,n}$	k	a	b
5		3	0	≥ 0	2
6	$m \ge 7$	divides $\epsilon_{7,8}$	1	≥ 1	
7	$m \ge 9$	divides $\epsilon_{9,11}$	2	≥ 2	
8		3	0	≥ 0	3
9		divides $\gcd(9, \epsilon_{9,9})$	1	≥ 0	
10	$m = 10$	multiple of 3	2	$= 0$	
	$m \ge 11$	divides $\epsilon_{7,8}$		≥ 1	
$11+3t$	$t \ge 0$	3	0	≥ 0	$4+t$
$12+3t$		divides $\gcd(9, \epsilon_{9,9})$	1		
$13+3t$			2		

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